Time Series lecture 5

SARIMA

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Lecture outline

1. Mean trends

2. Seasonal modelling: SARMA

3. ARIMA

Mean trends

Trend definition

Often an observed signal exhibits a trend. This is a tendency to increase or decrease over time. There may also be fluctuations over time. This model is given by

$$X_t = \mu_t + Y_t,$$

where μ_t is a time-dependent mean, and Y_t is a stationary process, for example $\mu_t = a + bt$.

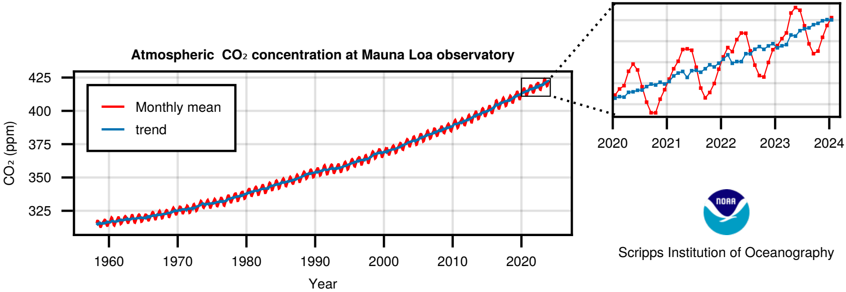

Figure: Example of a time series with a trend and seasonality.

Linear trend removal

There are two common approaches to trend adjustment:

1. Estimate a and b and remove the trend. We could use least squares regression, and then further analyse the residuals.
2. Take first differences, writing the difference operator $\nabla = (\mathrm{I} - \mathrm{B})$:

$$\begin{aligned} \nabla X_t &= X_t - X_{t-1} \\ &= a + bt + Y_t - (a + b(t-1) + Y_{t-1}) \\ &= b + Y_t - Y_{t-1} \\ &= b + \nabla Y_t. \end{aligned}$$

Thus we have got rid of the trend but we are left with the constant b, and also ∇Y_t rather than Y_t.

- In fact, the first difference of a stationary process is stationary, so if Y_t was stationary then so is ∇Y_t.
- If we difference again then we arrive at

$$\begin{aligned}\nabla^2 X_t &= \nabla(b + \nabla Y_t)\\ &= \nabla Y_t - \nabla Y_{t-1}\\ &= (Y_t - Y_{t-1}) - (Y_{t-1} - Y_{t-2})\\ &= Y_t - 2Y_{t-1} + Y_{t-2}.\end{aligned}$$

Thus the effect of μ_t has been completely removed from the observations.

- Let $X_t = Y_t + \mu_t$ where
  - Y_t is stationary,
  - μ_t is a $(q-1)^{\text{th}}$ degree polynomial.
- If μ_t is a $(q-1)^{\text{th}}$ degree polynomial in t then the q^{th} difference of μ_t will be zero. Therefore

$$\begin{aligned}\nabla^q X_t &= \nabla^q Y_t + \nabla^q \mu_t \\ &= \sum_{k=0}^{q} \binom{q}{k} (-1)^k Y_{t-k},\end{aligned}$$

Because ∇ is a linear operator.

Periodic trend removal

Sometimes we assume

$$X_t = s_t + Y_t,$$

and s_t is the periodic deterministic function. Y_t is assumed a zero-mean stationary process.

Periodic and linear trend removal

Let $\{Y_t\}$ be a stationary process with mean zero and let a and b be fixed constants. X_t is given by

$$X_t = a + bt + s_t + Y_t,$$

where s_t is a deterministic function with period 12, i.e. $s_{t+12} = s_t$. By forming the seasonal difference $\nabla^{(s)} X_t = X_t - X_{t-s}$ as well as the standard difference $\nabla X_t = X_t - X_{t-1} = \nabla^{(1)} X_t$, we can form $\nabla \nabla^{(12)} X_t$. Let us show that this is weakly stationary.

- First note that

$$\begin{aligned}\nabla^{(12)}X_t &= \nabla^{(12)}bt + \nabla^{(12)}Y_t \\ &= bt - b(t-12) + \nabla^{(12)}Y_t = 12b + \nabla^{(12)}Y_t \\ \nabla\nabla^{(12)}X_t &= \nabla\nabla^{(12)}Y_t \\ &= \nabla(Y_t - Y_{t-12}) = Y_t - Y_{t-1} - Y_{t-12} + Y_{t-13}. \qquad (5.1)\end{aligned}$$

- We can calculate

$$\mathbb{E}\left[\nabla\nabla^{(12)}X_t\right] = 0$$

$$\begin{aligned}\text{Cov}\left(\nabla\nabla^{(12)}X_t, \nabla\nabla^{(12)}X_{t-\tau}\right) = {} & \gamma_\tau - \gamma_{\tau+1} - \gamma_{\tau+12} + \gamma_{\tau+13} \\ & - \gamma_{\tau-1} + \gamma_\tau + \gamma_{\tau+11} - \gamma_{\tau+12} \\ & - \gamma_{\tau-12} + \gamma_{\tau-11} + \gamma_\tau - \gamma_{\tau+1} \\ & + \gamma_{\tau-13} - \gamma_{\tau-12} - \gamma_{\tau-1} + \gamma_\tau .\end{aligned}$$

This is a function of τ only and so we have a weakly stationary process.

Interaction between periodic and linear trend components

- Now instead set

$$X_t = (a + bt)s_t + Y_t,$$

where s_t is a deterministic function with period 12, i.e. $s_{t+12} = s_t$. Now instead show that $\nabla^{(12)2} X_t$ is stationary. We find that

$$\begin{aligned}\nabla^{(12)} X_t &= (a + bt)s_t + Y_t - (a + b(t-12))s_{t-12} + Y_{t-12} \\ &= 12bs_{t-12} + Y_t - Y_{t-12}\end{aligned}$$

$$\begin{aligned}\nabla^{(12)}\nabla^{(12)} X_t &= 12bs_{t-12} + Y_t - Y_{t-12} - (12bs_{t-24} + Y_{t-12} - Y_{t-24}) \\ &= Y_t - 2Y_{t-12} + Y_{t-24}\end{aligned}$$

The expectation of this is clearly zero, and the covariance is clearly just a function of $|\tau|$.

Seasonal modelling: SARMA

Seasonality

Many econometric and financial processes have seasonal features ($s = 4$ for quarterly data, $s = 12$ for monthly data etc). Consider the following model:

$$X_t = \varepsilon_t - \theta \varepsilon_{t-s}$$

- We can derive the autocorrelation sequence for this process

$$\begin{aligned}
\gamma_\tau &= \operatorname{Cov}\left(X_t, X_{t+\tau}\right) \\
&= \operatorname{Cov}\left(\varepsilon_t - \theta\varepsilon_{t-s}, \varepsilon_{t+\tau} - \theta\varepsilon_{t+\tau-s}\right) \\
&= \sigma_\varepsilon^2 \delta_\tau - \theta\sigma_\varepsilon^2 \delta_{\tau-s} - \theta\sigma_\varepsilon^2 \delta_{\tau+s} + \theta^2 \sigma_\varepsilon^2 \delta_\tau \\
&= \sigma_\varepsilon^2 \left(\left(1+\theta^2\right)\delta_\tau - \theta\left(\delta_{\tau-s} + \delta_{\tau+s}\right)\right)
\end{aligned}$$

- This is a seasonal moving average of order 1.

Definition 5.1 (Seasonal moving average)

A general seasonal moving average process $\text{SMA}(Q)_s$ takes the form

$$X_t = \varepsilon_t - \sum_{j=1}^{Q} \theta_j^{(s)} \varepsilon_{t-js}$$

where $\{\varepsilon_t\}$ is a mean-zero white noise process.

- We have the seasonal MA polynomial given by

$$\Theta^{(s)}(z) = 1 - \sum_{j=1}^{Q} \theta_j^{(s)} z^{sj}.$$

- So we could write

$$X_t = \Theta^{(s)}(\mathrm{B})\varepsilon_t.$$

Definition 5.2 (Seasonal autoregression)

A general seasonal autoregressive process SAR$(P)_s$ takes the form

$$X_t = \varepsilon_t + \sum_{j=1}^{P} \phi_j^{(s)} X_{t-js}$$

where $\{\varepsilon_t\}$ is a mean-zero white noise process.

- We have the seasonal AR polynomial given by

$$\Phi^{(s)}(z) = 1 - \sum_{j=1}^{P} \phi_j^{(s)} z^{sj}.$$

- So we could write

$$\Phi^{(s)}(\mathrm{B}) X_t = \varepsilon_t.$$

Definition 5.3 (SARMA)

We can combine the SAR and SMA with a standard ARMA to get a SARMA$(p, q) \times (P, Q)_s$, i.e.

$$\Phi(\mathrm{B})\Phi^{(s)}(\mathrm{B})X_t = \Theta(\mathrm{B})\Theta^{(s)}(\mathrm{B})\varepsilon_t$$

where $\{\varepsilon_t\}$ is a mean-zero white noise process.

- We have the following relations
  1. SMA$(Q)_s$ is SARMA$(0, 0) \times (0, Q)_s$.
  2. SAR$(P)_s$ is SARMA$(0, 0) \times (P, 0)_s$.
- Example: SARMA$(0, 1) \times (0, 1)_{12}$:

$$X_t = (1 - \theta\mathrm{B})\left(1 - \theta^{(12)}\mathrm{B}^{12}\right)\varepsilon_t.$$

ARIMA

Motivation

Consider starting from the AR(1) process

$$X_t = \phi X_{t-1} + \varepsilon_t.$$

- Stationarity of this process depends on $|\phi| < 1$.
- If we take $\phi = 1$ then the process is a random walk (which is not stationary):

$$X_t = X_{t-1} + \varepsilon_t$$

- However, the difference $\nabla X_t = \varepsilon_t$ is stationary.

Definition 5.4

We say that a process is an ARIMA(p,d,q) if the d^{th} difference is an ARMA(p,q). In other words

$$\Phi(\mathrm{B})(1-\mathrm{B})^d Y_t = \Theta(\mathrm{B})\varepsilon_t$$

- If we take $Z_t = \nabla^d Y_t$ then

$$Z_t = \sum_{j=1}^{p} \phi_j Z_{t-j} + \varepsilon_t - \sum_{j=1}^{q} \theta_j \varepsilon_{t-j}$$

- The first popular model we will study is the simple IMA $(1, 1)$ model. We take Z_t as the difference $Y_t - Y_{t-1}$,

$$Z_t = \varepsilon_t - \theta\varepsilon_{t-1}$$

- Then recalling the link between Y_t and Z_t we may write this as

$$Y_t = Y_{t-1} + \varepsilon_t - \theta\varepsilon_{t-1}$$

- This is NOT a stationary process.
- Assuming the process starts at $t = 0$ at $Y_0 = 0$ then

$$\begin{aligned} Y_1 &= 0 + \varepsilon_1 - \theta\varepsilon_0 \\ Y_2 &= Y_1 + \varepsilon_2 - \theta\varepsilon_1 \\ &= 0 + \varepsilon_1 - \theta\varepsilon_0 + \varepsilon_2 - \theta\varepsilon_1 \\ &= \cdots \\ Y_t &= \varepsilon_t + (1 - \theta) \sum_{j=1}^{t-1} \varepsilon_j - \theta\varepsilon_0. \end{aligned}$$

From this equation we can now find the variance of the process of interest:

$$\begin{aligned}\mathrm{Var}\left(Y_t\right) &= \sigma_\varepsilon^2 + (1-\theta)^2 \sum_{j=1}^{t-1} \sigma_\varepsilon^2 + \theta^2 \sigma_\varepsilon^2 \\ &= \sigma_\varepsilon^2 \left\{1 + (1-\theta)^2 (t-1) + \theta^2\right\}\end{aligned}$$

and the covariances (for $\tau > 0$) can also be calculated from this:

$$\mathrm{Cov}\left(Y_t, Y_{t+\tau}\right) = \sigma_\varepsilon^2 \left\{1 - \theta + (1-\theta)^2 (t-1) + \theta^2\right\}.$$

As $\tau / t \to 0$, $\mathrm{Corr}\left(Y_t, Y_{t+\tau}\right)$ tends to 1.

- The ARI$(1,1)$ process is defined from a stationary AR process.
- The ARI$(1,1)$ process is defined as ($|\phi| < 1$),

$$Z_t = \phi Z_{t-1} + \varepsilon_t$$

- Then recalling the link between Y_t and Z_t we may write this as

$$Y_t - Y_{t-1} = \phi\left(Y_{t-1} - Y_{t-2}\right) + \varepsilon_t$$

- Again assuming the process starts at $t = 0$ with $Y_0 = 0$ then

$$\begin{aligned}
Y_1 &= \varepsilon_1 \\
Y_2 &= \varepsilon_2 + (1+\phi)\varepsilon_1 \\
\ldots &= \ldots \\
Y_t &= \sum_{j=1}^{t} \sum_{i=0}^{t-j} \phi^i \varepsilon_j = \sum_{j=1}^{t} \frac{1-\phi^{t-j+1}}{1-\phi} \varepsilon_j
\end{aligned}$$

- We use this to determine the second order properties of the process.

$$\mathrm{Var}\left(Y_t\right) = \sigma_\varepsilon^2 \sum_{j=1}^{t} \left(\frac{1-\phi^{t-j+1}}{1-\phi} \right)^2 .$$

- We can also determine that when $\tau > 0$

$$\mathrm{Cov}\left(Y_t, Y_{t+\tau}\right) = \sigma_\varepsilon^2 \sum_{j=1}^{t} \left(\frac{1-\phi^{t-j+1}}{1-\phi} \right) \left(\frac{1-\phi^{t+\tau-j+1}}{1-\phi} \right)$$

- Again the correlation will be near to unity.